010_011 Post-Quantum-23

Let's change a little the initial system of 4 equations by adding to the right side numbers -1, 1, or 0 at random.

```
1*x + 2*y + 3*z = 14      (1)        1*x + 2*y + 3*z = 14 - 1        1*x + 2*y + 3*z = 13 = s1e
2*x + 1*y + 2*z = 10      (2)        2*x + 1*y + 2*z = 10 + 0        2*x + 1*y + 2*z = 10 = s2e
3*x + 2*y + 2*z = 13      (3)        3*x + 2*y + 2*z = 13 +1         3*x + 2*y + 2*z = 14 = s3e
3*x + 3*y + 1*z = 12      (4)        3*x + 3*y + 1*z = 12 - 1        3*x + 3*y + 1*z = 11 = s4e
```

Then we introduce the erroneous vector for the system corresponding to (1), (2), (3) equations denoting it by s123e,
and the erroneous vector for the system corresponding to (2), (3), (4) equations denoting it by s234e.
The corresponding matrices M123 and M234 are the same.

Then the solution of the first system of equations can be found by computing the inverse matrix to the matrix M123 we denote by M123i, and
the solution of the second system of equations can be found by computing the inverse matrix to the matrix M234 we denote by M234i

```
>> M123=[1 2 3; 2 1 2; 3 2 2]        >> s123e=[13;10;14]
M123 =                                s123e =
  1   2   3        x                   13
  2   1   2        y                   10
  3   2   2        z                   14

>> M123i=inv(M123)
M123i =
 -0.4000   0.4000   0.2000
  0.4000  -1.4000   0.8000
  0.2000   0.8000  -0.6000

>> w123e=M123i*s123e
w123e =
 1.6000
   2.4000
   2.2000
```

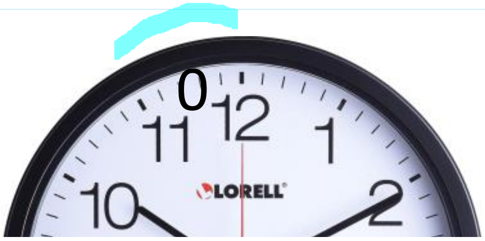
```
>> M234=[2 1 2; 3 2 2; 3 3 1]        >> s234e=[10;14;11]
M234 =                                s234e =
  2   1   2        x                   10
  3   2   2        y                   14
  3   3   1        z                   11

>> M234i=inv(M234)
M234i =
  -4   5  -2
   3  -4   2
   3  -3   1

>> w234e=M234i*s234e
w234e =
 6.0000e+00
  -2.0000e+00
   7.1054e-15
```

As we see solutions differs.

Learning With Errors - LWE Encryption

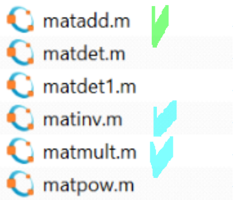


Time is computed modulo 12.
We will perform computations modulo 11.

In classical cryptography computations are performed modulo ***p***, when ***p*** is a large prime number.

In classical cryptography computations are performed modulo ***p***, when ***p*** is a large prime number.
Let ***p*** is prime then for all positive integers ***z***, expression ***z*** mod ***p*** is a reminder of ***z*** divided by ***p***.

Let ***p***=11.
Then computations of equations (1), (2), (3) and equations (2), (3), (4) we perform modulo 11.
Matrices M123, M234 and s123, s234 we leave the same and operations we perform modulo 11.
Matrix operations mod p are realized in .m files.
Library of .m files



matadd.m ✓
matdet.m
matdet1.m
matinv.m ✓
matmult.m ✓
matpow.m

We are using functions **matinv.m** and **matmult.m**
To find inverse matrix **Mi** mod ***p*** to matrix **M** we use
>> Mi=matinv(M,p)
To multiply inverse matrix **Mi** with vector **s** mod ***p*** to obtain vector **w** we use command
>> w=matmult(Mi,s,p)

Matrix computations 7z (zipped) .m files are presented in my Google drive:
https://drive.google.com/file/d/1l6T7FJ43ehD6BOuu08GRFmidd4H4_BEo/view?usp=sharing
Please unzip them and include in your Octave library.

Let's **again** change a little the initial system of 4 equations by adding to the right side the numbers -1, 1, or 0 at random.

Initial equations		Equations with errors	Final equations with errors
1*x + 2*y + 3*z = 14	(1)	1*x + 2*y + 3*z = 14 - 1	1*x + 2*y + 3*z = 13 = s1e
2*x + 1*y + 2*z = 10	(2)	2*x + 1*y + 2*z = 10 - 1	2*x + 1*y + 2*z = 9 = s2e
3*x + 2*y + 2*z = 13	(3)	3*x + 2*y + 2*z = 13 + 1	3*x + 2*y + 2*z = 14 = s3e
3*x + 3*y + 1*z = 12	(4)	3*x + 3*y + 1*z = 12 + 1	3*x + 3*y + 1*z = 13 = s4e

During the exam the system of 5 equations will be presented as Public Parameter - **PP**.
For encryption you should randomly select 2 or 3 equations of 5.
The number of variants is 10 + 10 = 20.

```
>> M123=[1 2 3; 2 1 2; 3 2 2]      >> s123e=[13;10;14]      >> M234=[2 1 2; 3 2 2; 3 3 1]      >> s234e=[10;14;11]
M123 =        w123e=                s123e =                 M234 =        w=234e                 s234e =
  1   2   3      x                    13                      2   1   2      x                       9
  2   1   2      y                    9                       3   2   2      y                       14
  3   2   2      z                    14                      3   3   1      z                       11

>> M123i=matinv(M123,p)                                       >> M234i=matinv(M234,p)
M123i =                                                       M234i =
  4   7   9                                                     7   5   9
```

```
>> M123i=matinv(M123,p)
M123i =
  4  7  9
  7  3  3
  9  3  6

>> I123=matmult(M123i,M123,p)
I123 =
  1  0  0
  0  1  0
  0  0  1

>> w123e=matmult(M123i,s123e,p)
w123e =
  6
  9
  0
```

```
>> M234i=matinv(M234,p)
M234i =
  7  5  9
  3  7  2
  3  8  1

>> I123=matmult(M123i,M123,p)
I123 =
  1  0  0
  0  1  0
  0  0  1

>> w234e=matmult(M234i,s234e,p)
w234e =
  8
  7
 10
```

As we see solutions differs as well since right sides of equations are corrupted by errors.
System of equations (1), (2), (3), (4) is **inconsistent** since if solution w123e satisfies one part of equations (1), (2), (3) it does not satisfies the other part of equations (2), (3), (4).

Exercises

Public parameters - **PP** is a system of 4 linear equations
with errors in the right-side computed mod 11:
1*x + 2*y + 3*z = 13 mod11 = 2 = s1e
2*x + 1*y + 2*z =   9 mod 11 = 9 = s2e
3*x + 2*y + 2*z = 14 mod 11 = 3 = s3e
3*x + 3*y + 1*z = 13 mod 11 = 2 = s4e

Learning with errors - LWE encryption is performed for a single bit ***b***={0, 1} at once.

The private key is a vector of unknown but **exact** wages per hour: **PrK** = **w'** = (x, y, z) = (1, 2, 3),
where **w'** is written as a row vector and is a transpose vector of vector column **w**.

To encrypt bit ***b***={0, 1} a random sub-system of equations is selected, e.g. let's take (1), (2), (3) sub-system of equations.
Then erroneous vector for the sub-system corresponding to (1), (2), (3) equations we denote by s123e.
The corresponding matrix M123 is the same.
The equations and erroneous vectors are of the form.

1*x + 2*y + 3*z = 2 = s1e (1)
2*x + 1*y + 2*z = 9 = s2e (2)
3*x + 2*y + 2*z = 3 = s3e (3)

```
>> M123=[1 2 3; 2 1 2; 3 2 2]
M123 =
  1  2  3      x
  2  1  2      y
  3  2  2      z
```

```
>> s123e=[13;10;14]
 s123e =
   2
   9
   3
```

Encryption

Let **Alice** is intending to encrypt bit ***b***={0, 1} and send it to **Bob.**
By having public parameters **PP**, consisting of system of 4 equations, **Alice** chooses **at random** some sub-system.
Let this sub-system consist of equations (1), (2), (3).
Having this sub-system, **Alice** adds (1), (2), (3) equations together mod 11:

$$+\begin{cases} 1*x + 2*y + 3*z = 2 = s1e \ \ (1) \\ 2*x + 1*y + 2*z = 9 = s2e \ \ (2) \\ 3*x + 2*y + 2*z = 3 = s3e \ \ (3) \end{cases}$$

$$\overline{6*x + 5*y + 7*z = 14 \bmod 11 = \mathbf{3}}$$

If **Alice** intends to encrypt bit ***b***=**0** then she sends original equation 6*x + 5*y + 7*z = **3** corresponding to ciphertext $\mathbf{c_0}$ to **Bob**.

If **Alice** intends to encrypt bit ***b***=**1** then she sends equation 6*x + 5*y + 7*z = **3** + **5** = **8** corresponding to ciphertext $\mathbf{c_1}$ to **Bob**.
This means that for ***b***=**1** encryption **Alice** adds to the right side of equation number **5**.
This number corresponds to the rounded down number **p**/2 = 11/2 = 5.5, i.e. it is equal to **5**.

Decryption

For decryption **Bob** is using his **PrK** = **w'** = (x, y, z) = (1, 2, 3).

Let **Bob** received ciphertext $\mathbf{c_0}$: 6*x + 5*y + 7*z = **3**.
Then **Bob** puts his **PrK** values (1, 2, 3) to the left side of $\mathbf{c_0}$ equation and obtains:

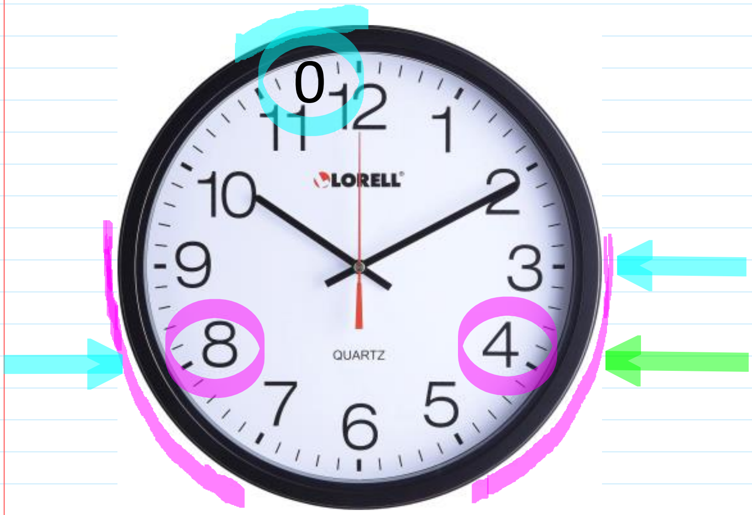
6*1 + 5*2 + 7*3 = 6 + 10 + 21 = 37 mod 11 = **4**

The number **4** is close to the right side number **3** of the received ciphertext $\mathbf{c_0}$: 6*x + 5*y + 7*z = **3**
This means that **Bob** decrypted bit ***b***=**0**.

Let **Bob** received ciphertext $\mathbf{c_1}$: 6*x + 5*y + 7*z = **8**.
Then **Bob** puts his **PrK** values (1, 2, 3) to the left side of $\mathbf{c_1}$ equation and obtains:

6*1 + 5*2 + 7*3 = 6 + 10 + 21 = 37 mod 11 = **4**

The number **4** is far from to the right side number **8** of the received ciphertext $\mathbf{c_1}$: 6*x + 5*y + 7*z = **8**.
The difference is **8** - **4** = **4** and it is more than the errors values fluctuating in the range [-1, 1].
This means that **Bob** decrypted bit ***b***=**1**.



6*x + 5*y + 7*z = **3** mod 11

Bob received vector row:

```
>> Esum=[6 5 7]
Esum =
   6   5   7
```

Bob received number **3** as a right side of equation

Bob inputs vector w as his **PrK** = **w'** = (x, y, z) = (1, 2, 3).

```
>> w=[1;2;3]
w =
   1
   2
   3
```

Bob computes Right Side (RS) of equation:

```
>> p=11
p = 11
>> RS0=matmult(Esum,w,p)
RS0 = 4
```

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